Wormholes with varying equation of state parameter

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Abstract

We propose wormholes solutions by assuming space dependent equation of state parameter. Our models show that the existence of wormholes is supported by arbitrary small quantities of averaged null energy condition (ANEC) violating phantom energy characterized by variable equation state parameter.

A Wormhole is a 'tunnel' through curved spacetime, connecting two widely separated regions of our Universe or even of different Universe. In a pioneer work, Morris and Thorne [1] observed, to hold a wormhole open, one has usually used an exotic matter, which violates the well known energy conditions. The exotic matter is a hypothetical form of matter that has either negative energy density or positive energy density which produces negative pressure / tension. In last few years, exotic matter has been becoming an active area of research in wormhole physics [2]. Since all known matters obey the null energy, $T_{\mu\nu}k^\mu k^\nu > 0$, where $T_{\mu\nu}$ is the energy stress tensor and k^μ any null vector, several authors [3] have considered scalar tensor theories to build wormhole like spacetime with the presence of ordinary matter in which scalar field may play the role of exotic matter. In an interesting paper, Vollick has shown how to produce exotic matter using scalar field [4]. Recent astrophysical observations indicated that the Universe at present is accelerating . There are different ways of evading these unexpected behavior . Most of these attempts focus on Alternative gravity theories or the supposition of existence of a hypothetical dark energy with a positive energy density and a negative pressure [5]. The matter with the property, energy density, $\rho > 0$ but pressure $p < 0$ is known as Phantom Energy. The Phantom Energy violates the null energy condition what is needed to support traversable wormholes. So phantom energy may play a possible role for constructing wormhole like spacetime.

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Several authors have recently discussed the physical properties and characteristics of traversable wormholes by taking Phantom Energy as source [6]. Recent observational analysis involving X-ray luminosity of galaxy clusters and SNe type Ia data suggest that we live in a flat Universe and its present acceleration stage driven by a dark energy component whose equation of state may evolve in time [7]. Several authors have studied cosmological models assuming variable equation of state parameter[8]. Since in the literature of wormhole physics, this dark energy component is known as phantom energy, in this article, we propose wormhole solutions supported by phantom energy where equation of state parameter is a function of radial coordinate rather a constant. In the present work, we consider several toy models of wormholes. We have established a matching of each interior wormhole metric with an exterior Schwarzschild metric.

We consider the model, which is characterized by the exotic equation of state,

$$\frac{p}{\rho} = -m(r) \tag{1}$$

where $m(r)$ is a positive function of radial coordinate.

A static spherically symmetric Lorentzian wormhole can be described by a manifold R^2XS^2 endowed with the general metric in Schwarzschild co-ordinates (t, r, θ, ϕ) as

$$ds^2 = -e^{2f(r)}dt^2 + \frac{1}{[1 - \frac{b(r)}{r}]}dr^2 + r^2 d\Omega_2^2 \tag{2}$$

where, $r\epsilon(-\infty, +\infty)$.

To describe a wormhole, the redshift function f (r) should be finite and the shape function obeys the following properties

$$b(r_0) = r_0 \tag{3}$$

where r_0 is the throat of the wormhole.

$$b'(r_0) < 1 \tag{4}$$

$$b(r) < r, r > r_0 \tag{5}$$

Also the spacetime is asymptotically flat i.e. $\frac{b(r)}{r} \to 0$ as $\mid r \mid \to \infty$.

According to Morris and Thorne [1], we assume $f = constant$, to make the problem simpler. This assumption implies that a traveller feels a zero tidal force. This supposition would help for an advanced engineer to construct a traversable passage.

Using the Einstein field equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$, in orthonormal reference frame (with $c = G = 1$) , we obtain the following stress energy scenario,

$$\rho(r) = \frac{b'}{8\pi r^2} \tag{6}$$

$$p(r) = \frac{1}{8\pi}[-\frac{b}{r^3}] \tag{7}$$

$$p_{tr}(r) = \frac{1}{8\pi}(1 - \frac{b}{r})[\frac{(-b'r + b)}{2r^2(r-b)}] \tag{8}$$

where $\rho(r)$ is the energy density, $p(r)$ is the radial pressure and $p_{tr}(r)$ is the transverse pressure.

Using the conservation of stress energy tensor $T^{\mu\nu}_{;\nu} = 0$, one can obtain the following equation

$$p' + \frac{2}{r}p - \frac{2}{r}p_{tr} = 0 \tag{9}$$

Now from equation (1), by using (6) and (7), one gets,

$$\frac{p}{\rho} = -m(r) = -\frac{b}{rb'} \tag{10}$$

Specialization 1 : $m(r) = k(constant)$**.**

Now consider the special case, $m(r) = k(constant)$, then equation (10) yields,

$$b = b_0 r^{\frac{1}{k}} \tag{11}$$

[b_0 is an integration constant]

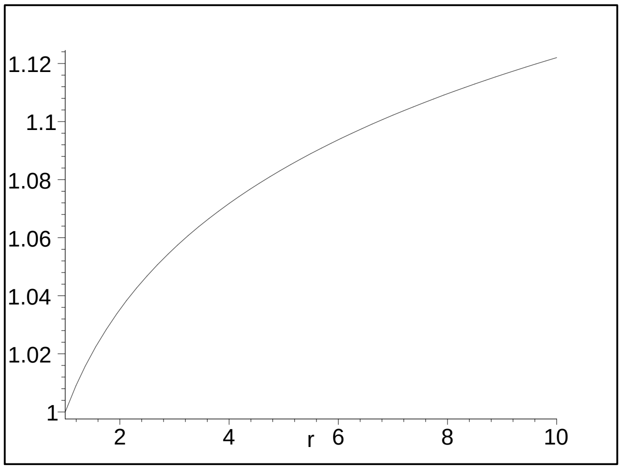

Figure 1: Diagram of the shape function of the wormhole

Since the spacetime is asymptotically flat i.e. $\frac{b(r)}{r} \to 0$ as $\mid r \mid \to \infty$, then the equation (11) is consistent only when $k > 1$.

The throat of the wormhole occurs at

$$r = r_0 = b_0^{\frac{k}{k-1}} \tag{12}$$

Now we match the interior wormhole metric to the exterior Schwarzschild metric . To match the interior to the exterior, we impose the continuity of the metric coefficients, $g_{\mu\nu}$, across a surface, S , i.e. $g_{\mu\nu(int)}|_S = g_{\mu\nu(ext)}|_S$.

[This condition is not sufficient to different space times. However, for space times with a good deal of symmetry (here, spherical symmetry), one can use directly the field equations to match [9]]

The wormhole metric is continuous from the throat, $r = r_0$ to a finite distance $r = a$. Now we impose the continuity of g_{tt} and g_{rr},

$g_{tt(int)}|_S = g_{tt(ext)}|_S$

$g_{rr(int)}|_S = g_{rr(ext)}|_S$

at $r = a$ [i.e. on the surface S] since $g_{\theta\theta}$ and $g_{\phi\phi}$ are already continuous. The continuity of the metric then gives generally

$e^{2f}{}_{int}(a) = e^{2f}{}_{ext}(a)$ and $g_{rr(int)}(a) = g_{rr(ext)}(a)$.

Hence one can find

$$e^{2f} = (1 - \frac{2GM}{a}) \tag{13}$$

and $1 - \frac{b(a)}{a} = (1 - \frac{2GM}{a})$ i.e. $b(a) = 2GM$

This implies

$b_0 a^{\frac{1}{k}} = 2GM$

Hence,

$$a = (\frac{2GM}{b_0})^k \tag{14}$$

i.e. matching occurs at $a = (\frac{2GM}{b_0})^k$.

The interior metric $r_0 < r \leq a$ is given by

$$ds^2 = -[1 - b_0 a^{\frac{1-k}{k}}]dt^2 + \frac{dr^2}{[1 - b_0 r^{\frac{1-k}{k}}]} + r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{15}$$

The exterior metric $a \leq r < \infty$ is given by

$$ds^2 = -[1 - \frac{b_0 a^{\frac{1}{k}}}{r}]dt^2 + \frac{dr^2}{[1 - \frac{b_0 a^{\frac{1}{k}}}{r}]} + r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{16}$$

Here, one can see that the null energy condition is violated, $p + \rho < 0$ and consequently all the other energy conditions. Now we will check whether the wormhole geometry is, in principle, suffered by arbitrary small amount averaged null energy condition (ANEC) violating phantom energy. The ANEC violating matter can be quantified by the integrals $I = \oint \rho dV$, $I = \oint (p_i + \rho) dV$. In the model, we have assumed that the ANEC violating matter is related only to p_r, not to the transverse components [as one can see from field equations (6) - (8), $p_{tr} = (\frac{-1+m(r)}{2})\rho$ and $m(r) > 1$].

According to Visser et al [10] , the information about the 'total amount' of ANEC violating matter in the spacetime is given by the integral,

$$I = \oint (p + \rho) dV = 2 \int_{r_0}^{\infty} (p + \rho) 4\pi r^2 dr \tag{17}$$

[$dV = r^2 \sin\theta dr d\theta d\phi$, factor two comes from including both wormhole mouths]

From the field equations, one can get,

$$p + \rho = \frac{1}{8\pi r}(1 - \frac{b}{r})[\ln \frac{1}{(1 - \frac{b}{r})}]' \tag{18}$$

Hence,

$$I = [(r - b)\ln(\frac{r}{r - b})]_{r_0}^{\infty} - \int_{r_0}^{\infty}[(1 - b')\ln(\frac{r}{r - b})]dr \tag{19}$$

Since at the throat r_0, $b(r_0) = r_0$, the boundary term at r_0 vanishes. The boundary term at infinity vanishes because the spacetime is asymptotically flat.

Then,

$$I = -\int_{r_0}^{\infty}[(1 - b')\ln(\frac{r}{r - b})]dr \tag{20}$$

Now we consider the wormhole field deviates from the throat out to a radius 'a'. Thus the total amount of ANEC violating matter is to matching the interior solution to an exterior spacetime at 'a'. Then the volume integral takes the value,

$$I = [-a + b(a)]\ln a + (a - r_0) - k[b(a) - b(r_0)] + [-a + b(a)][\ln(-a + b(a)) - 1] \tag{21}$$

Taking the limit $a \rightarrow r_0$, one can note that, $I = \oint(p + \rho)dV \rightarrow 0$.

Thus our model represents wormhole with arbitrarily small quantities of ANEC violating phantom energy. Hence an advanced engineer may construct a wormhole taking vanishing amount of the phantom energy material.

Specialization 2 : Specific shape function : $b(r) = D(1 - \frac{A}{r})(1 - \frac{B}{r})$**.**

Consider the specific form of the shape function as

$$b(r) = D(1 - \frac{A}{r})(1 - \frac{B}{r}) \tag{22}$$

where A, B and D(> 0) are arbitrary constants.

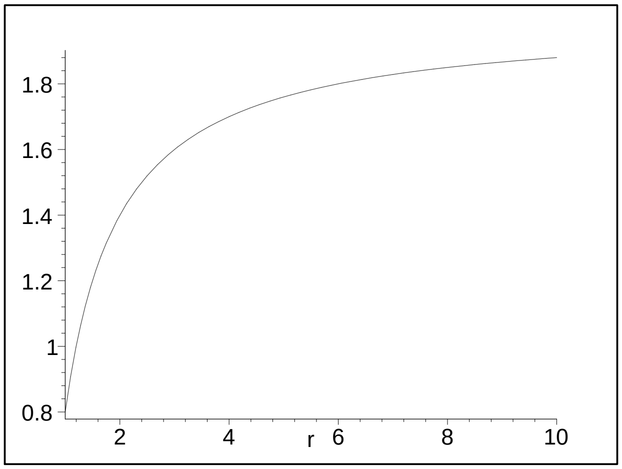

Figure 2: Diagram of the shape function of the wormhole

For this case, the equation of state parameter function takes the form

$$m(r) = \frac{[(r-A)(r-B)]}{[(A+B)r - 2AB]} \tag{23}$$

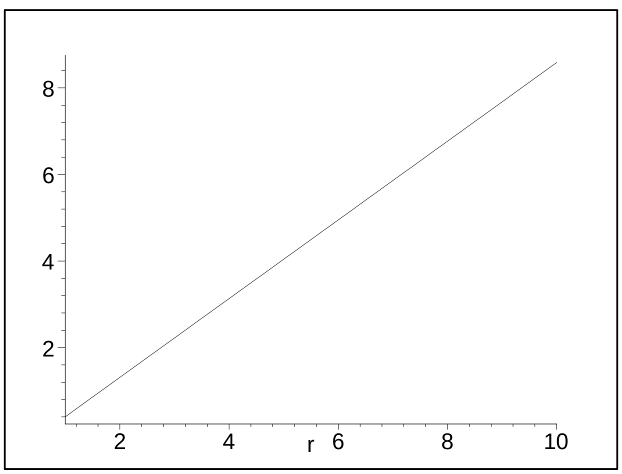

Figure 3: Diagram of the Equation of state parameter

We will now verify whether the particular choice of the shape function would represent the wormhole structure. One can see easily that $\frac{b(r)}{r} \to 0$ as $\mid r \mid \to \infty$.

Throat of the wormhole occurs at $r = r_0$, where r_0 satisfies the following equation $b(r_0) = r_0$ i.e. $r_0^3 - Dr_0^2 + (A+B)Dr_0 - ABD = 0$.

The solution of this equation is

$$r_0 = S + T + \frac{D}{3} \tag{24}$$

where ,

$S = [R + \sqrt{Q^3 + R^2}]^{\frac{1}{3}}$ and $T = [R - \sqrt{Q^3 + R^2}]^{\frac{1}{3}}$,

$Q = \frac{3D(A+B)-D^2}{9}, R = \frac{27ABD+2D-9D^2(A+B)}{54}$

Since r_0 is a root of the above equation , then by standard theorem of algebra, either $g(r) \equiv b(r) - r < 0$ for $r > r_0$ and $g(r) > 0$ for $r < r_0$ or $g(r) > 0$ for $r > r_0$ and $g(r) < 0$ for $r < r_0$. Let us take the first possibility and one can note that for $r > r_0$, $g(r) < 0$, in other words, $b(r) < r$. But when $r < r_0$, $g(r) > 0$, this means, $b(r) > r$, which violates the wormhole structure given in equation(2).

Now we are interested to match interior wormhole metric to the exterior Schwarzschild metric. Since the wormhole metric is continuous from the throat $r = r_0$, to a finite distance $r = a$, we use the continuity of the metric coefficients across a surface, as above, i.e.

$g_{tt(int)}|_S = g_{tt(ext)}|_S$

$g_{rr(int)}|_S = g_{rr(ext)}|_S$

at $r = a$.

Hence one can find,

$e^\nu = (1 - \frac{2GM}{a})$

and $b(a) = 2GM$

This implies matching occurs at $a = \frac{AD+BD+\sqrt{(AD+BD)^2-4ABD(D-2GM)}}{2(D-2GM)}$.

Here the interior metric $r_0 < r \leq a$ is given by

$$ds^2 = -[1 - \frac{D}{a}(1 - \frac{A}{a})(1 - \frac{B}{a})]dt^2 + \frac{dr^2}{[1 - \frac{D}{r}(1 - \frac{A}{r})(1 - \frac{B}{r})]} + r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{25}$$

The exterior metric $a \leq r < \infty$ is given by

$$ds^2 = -[1 - \frac{D}{r}(1 - \frac{A}{a})(1 - \frac{B}{a})]dt^2 + \frac{dr^2}{[1 - \frac{D}{r}(1 - \frac{A}{a})(1 - \frac{B}{a})]} + r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{26}$$

In this case, we measure the total amount of ANEC violating matter in the spacetime with a cut off of the stress energy at 'a' as

$$I = [a-b(a)][1+\ln(1-\frac{b(a)}{a})]+(a-r_0)-D\ln\frac{a}{r_0}-D(A+B)(\frac{1}{a}-\frac{1}{r_0})-2ABD(\frac{1}{a^2}-\frac{1}{r_0^2}) \quad (27)$$

By taking the limit $a \to r_0$, one readily verifies that, $I \to 0$. This proves that it is possible to construct wormhole with arbitrarily small amount of phantom energy characterized by variable equation of state parameter.

According to Morris and Thorne [1] , the 'r' co-ordinate is ill-behaved near the throat, but proper radial distance

$$l(r) = \pm \int_{r_0^+}^{r} \frac{dr}{\sqrt{1-\frac{b(r)}{r}}} \quad (28)$$

must be well behaved everywhere i.e. we must require that $l(r)$is finite throughout the space-time .
For our model, (taking $B = 0$), one can determine the proper distance through the wormhole as

$$l(r) = \sqrt{r^2 - Dr + DA} + \frac{D}{2}\ln[\frac{2\sqrt{r^2 - Dr + DA} + 2r - D}{2r_0 - D}] \quad (29)$$

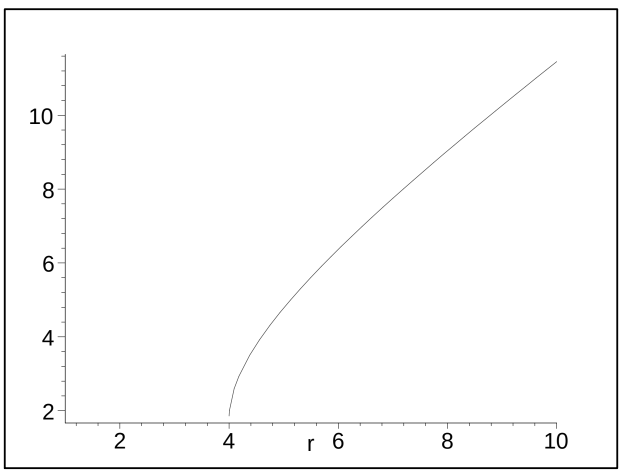

Figure 4: Diagram of the radial proper distance (D = 2, A = - 4, $r_0 = 4$)

The radial proper distance is measured from r_0 to any $r > r_0$. Note that on the throat $r = r_0$, $l = 0$.

Specialization 3 : Specific shape function : $b(r) = A \tanh Cr$**.**

Now we make the specific choice for the shape function as

$$b(r) = A \tanh Cr \tag{30}$$

where A (> 0) and C are arbitrary constants.

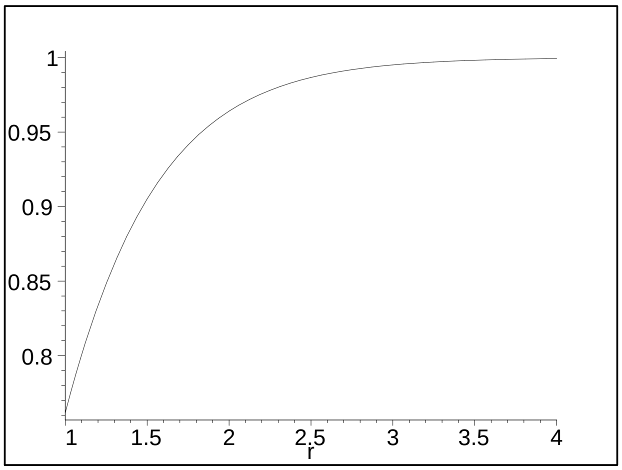

Figure 5: Diagram of the shape function of the wormhole

Using the equation (10), one gets

$$m(r) = \frac{C}{2r} \sinh 2Cr \tag{31}$$

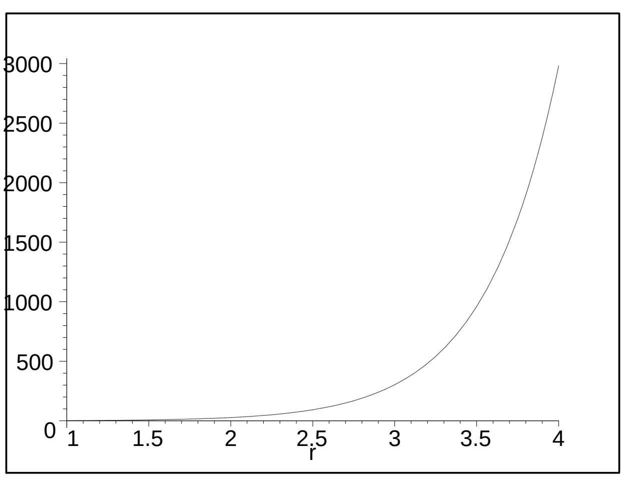

Figure 6: Diagram of the Equation of state parameter

It is easy to verify the above particular choice of the shape function would represent wormhole structure. Here, $\frac{b(r)}{r} \to 0$ as $\mid r \mid \to \infty$ and throat occurs at $r = r_0$ for which $b(r_0) = r_0$ i.e. $A \tanh Cr_0 = r_0$.

[If one chooses $A = 2$ and $C = 1$, the graph of the function $F(r) = b(r) - r$ indicates the point r_0 where $F(r)$ cuts the 'r' axis. From the graph, one can also note that when $r > r_0$, $F(r) < 0$ i.e. $b(r) - r < 0$. This implies $\frac{b(r)}{r} < 1$ when $r > r_0$.]

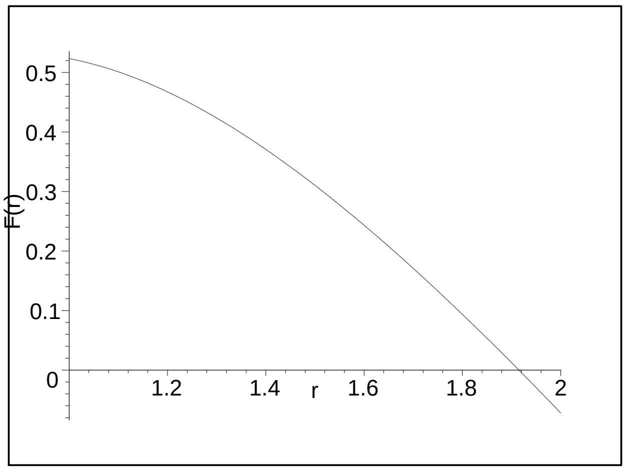

Figure 7: Throat occurs where $F(r)$ cuts 'r' axis

Now matching this interior metric of wormhole with the exterior Schwarzschild metric at a, where $a = \frac{1}{2C} \ln \frac{A+2GM}{A-2GM}$, one gets, the interior metric $r_0 < r \leq a$ as

$$ds^2 = -[1 - \frac{A \tanh Ca}{a}]dt^2 + \frac{dr^2}{[1 - \frac{A \tanh Cr}{r}]} + r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{32}$$

Here the exterior metric $a \leq r < \infty$ is given by

$$ds^2 = -[1 - \frac{A \tanh Ca}{r}]dt^2 + \frac{dr^2}{[1 - \frac{A \tanh Ca}{r}]} + r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{33}$$

In this case, the amount of ANEC violating matter in the spacetime with a cut off of the stress energy at 'a' is given by

$$I = [a - A \tanh Ca] \ln a + (a - r_0) - A[C(a - r_0) - \frac{c^3}{9}(a^3 - r_0^3) + \frac{2c^5}{75}(a^5 - r_0^5) - ..... \frac{(-1)^{n-1} B_n 2^{2n}(2^{2n}-1)}{(2n-1)(2n!)} c^{2n-1}(a^{2n-1} - r_0^{2n-1}) + ...] + (a - A \tanh Ca) \ln[(a - A \tanh Ca) - 1]$$

where, B_n is the Bernoulli number.

If one takes $a \to r_0$, then $I \to 0$. In other words, this type of wormhole can be constructed with arbitrary small quantities of ANEC violating phantom energy material.

In conclusion, our aim in this paper to provide a prescription for obtaining wormhole where stress energy tensor is characterized by phantom energy with variable equation of state parameter. As mentioned above, to be a wormhole solution, the condition $b'(r_0) < 1$ is imposed. Now for the first case, $b'(r_0) = \frac{1}{k} < 1$, since $k > 1$, corresponding to solution(11). For the second case, $b'(r_0) < 1$ if $r_0 > (A+B) \pm \sqrt{(A+B)^2 - 3AB}$, corresponding to solution(22) and for the last case, one has to assume $r_0 > \frac{1}{C} \cosh^{-1} \sqrt{AC}$, corresponding to the solution (30). We have established a matching each of three interior wormhole metrics with the exterior Schwarzschild metric. Our models reveal the fact that one may construct wormholes with arbitrarily small amounts of phantom energy characterized by variable equation of state parameter. The effective mass inside the radius 'r' is defined by $M(r) = \frac{b(r)}{2}$ and the limit, $\lim_{r\to\infty} M(r) = M$, if exists, represents the asymptotic wormhole mass seen by an distant observer. In the first case, this limit does not exist whereas for the last two cases, one can see that M exists and equal to D for the second case and equal to A for the last case. This implies that a distant observer could not see any difference of gravitational nature between Wormhole and a compact mass 'M'.

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